ON ASYMMETRY AXES AND AN INVARIANT OF THE OVAL OF DESCARTES

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ABCTRACT

We define the inner(+) and the outer(-) part of the Cartesian Oval as $mr_1 \pm nr_2 = kc$ on bipolar coordinates.
We can consider on Minor axis(asymmetry axis) of the inner part of the Oval, and can define Major axis (asymmetry axis) of the outer part of the Oval. This major axis is a segment which connects the middle point O of symmetry axis and the point Fp on the Oval, which is at the longest distance from the point O. Then, the length of major axis is ao*(1+eL*eR)^(1/2) (where ao is a half of the length of the symmetry-axis, eL, eR are left and right eccentricity of the Oval, respectively.) And, we can say that Cardioid is the special case of Cartesian Oval. In this case, the length of the major axis is ao*2^(1/2). Moreover, we have found the following Lemma.
[Lemma] Let bi be the length of Minor axis of the inner part of the Oval, let ai and ao be the half length of symmetry-axis of the inner and outer part, respectively. Let bo be the length of the Major axis of the outer part. Then, the following invariant holds.
(bi/ai)^2+(bo/ao)^2=2
Keywords: Oval, Outer part, Major axis, Cardioid, Invariant.

1. INTRODUCTION

The Oval of Descartes is an extension of ellipse, and has a much more variety of classical geometric theorem in comparison with other ovals reported by EBISUI(1995a) as the convex closed curves. Among them, we have found and reported a concept corresponding to the minor axis of an ellipse. Consideration was limited at that time to the inner part in which the Cartesian Oval becomes so called convex one. The Cartesian Oval is a biquadratic and doubly closed curve in the xy coordinate system, and has a property as a union of its inner and outer parts in addition, so we defined this time the concept corresponding to the minor axis in the outer part, too, and considered its properties. New properties of the minor axis is also considered in addition. Besides, the outer part also becomes the curve called Cardioid in a special case. We thought about this special example to some extent. Last, we found out a metric invariant in regard of the axes of the both inner and outer parts of the biquadratic curve, the oval, so it is reported herewith.

2. DEFINITION OF OUTER PART OF THE OVAL

Definition of the oval outer part may be said similar to that of the inner part in the previous discussion mentioned by EBISUI(1994), but it is again described below in notice only of the outer part.
[Definition 1 of Outer Part]
It stands for 『the curve, having a constant ratio of distances from a fixed circle and a fixed point within the fixed circle, and being outside of the fixed circle』 where the distance from a fixed circle stands for the shortest distance which connects the point and a point on the circle. In short, the segment, which is the distance between the point and fixed circle, can also be said locating on a normal of the fixed circle.

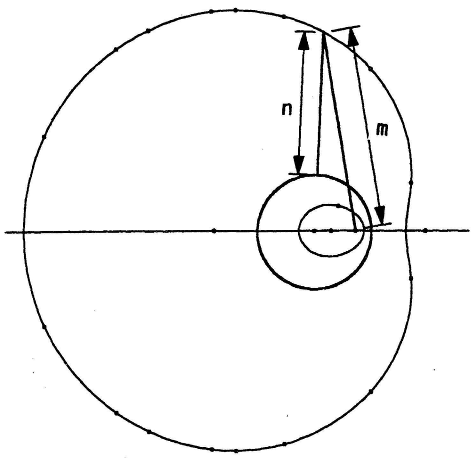

Fig. 1 Definition 1 of the Oval

[Definition 2 of Outer Part]

It is expressed with the following equation in the bipolar coordinates: $mr_1-nr_2=kc$ (1)
Here, r_1 and r_2 denote distances from the two poles to the curve, and c is the distance between the both poles. m, n, k are arbitrary constants with condition $k>m>n>0$.

[Definition 3 of Outer Part]
As shown in Fig. 2, 『Two fixed points are taken on a radius of a fixed circle O. Intersectons of mutually parallel lines l_1 and l_2 which passes these two fixed points (F_1 and F_2) with the circumference are denoted as M, M', N and N'. Here, taken on the extension of the line MN is the point Q with which $ON//F_1Q$ holds true. (At this time $OM//F_2Q$ holds true from Pappos' theorem). Then, when one lets the straight lines l_1 and l_2 make turn round F_1 and F_2, and obtains a sequence of points of Q by means of similar drawings, it draws the outer part of the Cartesian Oval』.

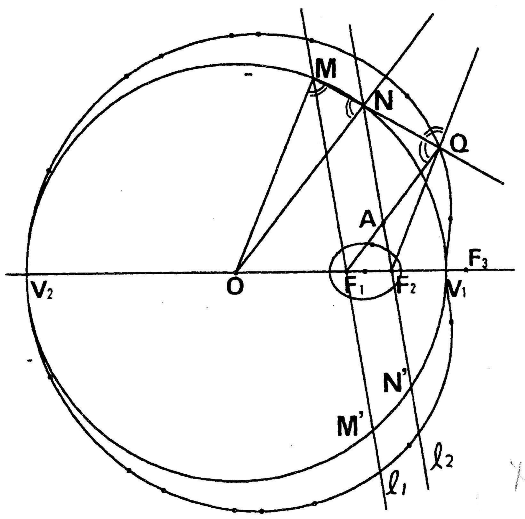

Fig. 2 Definition 3 of the Oval

[Proof of Definition 3 being equivalent to Definition 2]
Provided $F_1F_2=c$ and $OF_1 : OF_2=n : m$ in Fig. 2, one can settle $ON=OM=kc/(m-n)$ for $m \neq n$ because an arbitrary constant k is included. Then $\triangle$ OMN is an isosceles triangle, and when the intersection of F_1Q and F_2N is denoted as A, bisector of the external angle Q of $\triangle AQF_2$ is QN from $ON//F_1Q$ and $OM//F_2Q$. Accordingly, $F_2Q : QA=F_2N : NA= OF_2 : OF_1= m : n$, leading

$$QA=(n/m)F_2Q \quad (2)$$

And, providing $F_1A=OM(m-n)/m=kc/m$ (3)

and $F_1Q=r_1,\ F_2Q=r_2$ (4)

from $F_1Q-QA = F_1A$ and Eqs. (2), (3), (4), we obtain

$$r_1-(n/m)r_2=kc/m \quad \therefore \quad mr_1-nr_2=kc$$

Now, it is also clear that the point Q exists outside the circle O and that the outer part is symmetric with respect to the straight line F_1F_2. It is further evident that the outer part circumscribes with the circle O at intersections V_1 and V_2 of the circle with line F_1F_2, also because the lines l_1 and l_2 move so as to overlop the line F_1F_2. To speak in inverse, the circle O inscribes the outer part of the Oval.
From the fact that radius of the circle O can be shown as $OM=kc/(m-n)$, in the mean time, the outer part is normalized with this length, giving the family of outer parts of the Cartesian Oval to be composed as shown in Fig. 3. Here, for the circle O, $OF_1/OV_1=(cn/(m-n))/(kc/(m-n))=n/k=e_L<1$, and similarly $OF_2/OV_1=m/k=e_R<1$, where e_L and e_R are the left and right eccentricities, respectively, in the outer part. Numerical values in parentheses of Fig. 3 are (e_L,e_R).

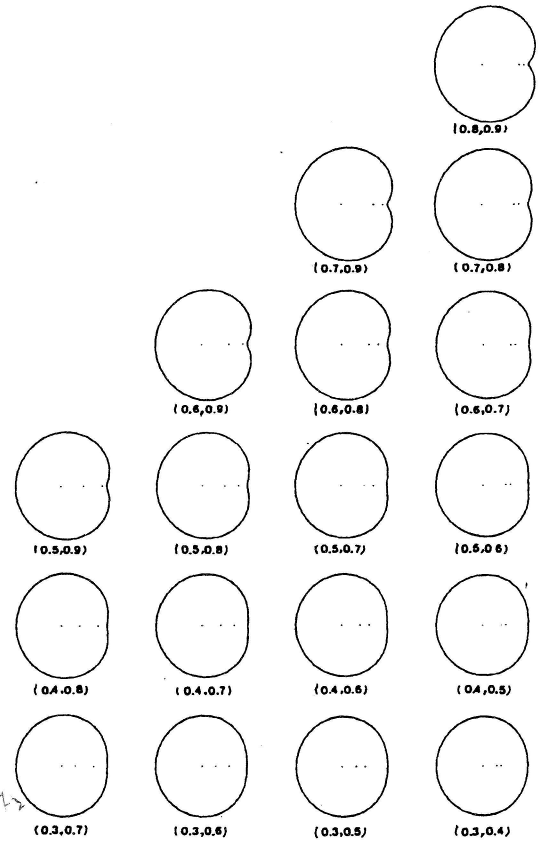

Fig. 3 Variety of Outer Parts

Now, let us call the line segment, V_1V_2, as the outer minor axis. This corresponds to the major axis of the inner part.

3. MAJOR AXIS OF OUTER PART

3.1 Definition of Major Axis of Outer Part

The outer minor axis of outer part was found in Section 2. Then the conception similar to the minor axis of the inner part is defined for the outer part.
The definition is, as shown in Fig.4, 『The major axis of the outer part of the Oval is the longest of line segments connecting the midpoint of its symmmetry axis and a point Q on the outer part』

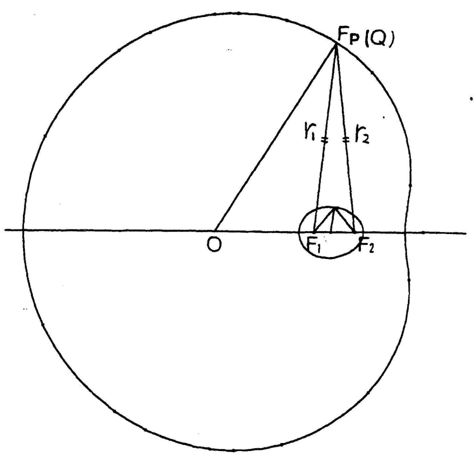

Fig. 4 Definition of Major axis (Outer major axis)

3.2 Position of Major Axis of Outer Part and Its Deduction

When the outer part has been defined with $mr_1-nr_2=kc$, we take the symmetry axis as x axis, its midpoint as the origin O(0,0), and the point Q on the outer part as Q (X,Y). Provided $\angle QF_1F_2=\theta$, we obtain

$$X^2+Y^2$$
$$=r_1^2+(nc/(m-n))^2-2r_1(nc/(m-n))\cos(\pi-\theta) \qquad (5)$$

from $OF_1=nc/(m-n)$ by means of the cosine theorem.
Besides, obtained in $\triangle QF_1F_2$ is

$$r_2^2=r_1^2+c^2-2r_1c\cos\theta \qquad (6)$$

from the cosine theorem.
Eliminating r_2 and θ from Eqs. (1), (5) and (6), we obtain

$$OQ^2=X^2+Y^2=-(m/n)(r_1-kc/(m-n))^2 + (k^2+mn)c^2/(m-n)^2$$

When r_1 is equal to $kc/(m-n)$ in the above equation which is the quadratic expression on r_1, OQ takes the maximum $\sqrt{(k^2+mn)}\,c/(m-n)$. If using e_L, e_R and $OM=ao=kc/(m-n)$ which is the radius of the circle O in the previous section, this becomes $bo=ao\sqrt{1+e_Le_R}$.
By the way, $ao=kc/(m-n)$ agrees with $r_1=kc/(m-n)$ which is the expression of defining the outer part ($mr_1-nr_2=kc$), for $r_1=r_2$.
Accordingly, 『Major axis of the Oval outer part (We call this the outer major axis) is the line segment which connects the center of symmetry axis with the point on the Oval (called the far point Fp) locating at the same distance from the focal points F_1 and F_2. Its length is $bo=ao\sqrt{1+e_Le_R}$.』

3.3 Properties of Outer Major Axis of Outer Part

Concerning the property similar to the minor axis of the Oval inner part in the literature (EBISUI, 1994), we list here these natures and figures only. By the way, the outer major axis is that of asymmetry as evident from the previous section.

[1] The outer major axis of the Oval outer part is located on a normal of the Oval at the far point (Fp). Fig. 5-1 shows a drawing method for a normal at a general position, and Fig. 5-2 shows its location and that outer major axis is on a normal line.

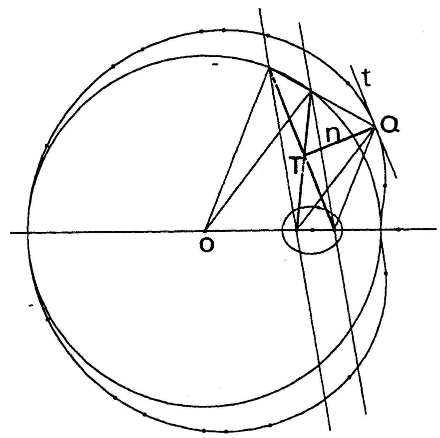

Fig. 5-1 Normal of Outer Part

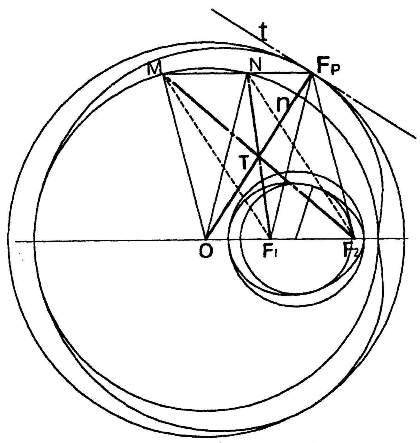

Fig. 5-2 Outer major axis is normal

[2] The end point (far point) on the Outer major axis is not the vertex in differential geometry. In short, the method of drawing the vertex by means of Definition 3 is expressed as shown in Fig. 6. Namely, when the line l_1 and the line F_1F_2 are perpendicular and $\cos\theta = m/k$, the vertex is determined. Besides, tangent at the vertex passes the third focal point.

[3] The outer part exists in the space between two concentric circles that is to say, its inscribed and circumscribed auxiliary circles. Refer to Fig. 7

[4] Perpendicular bisector of the outer major axis passes the third focal point. Refer to the literature (EBISUI, 1995b) and Fig. 7

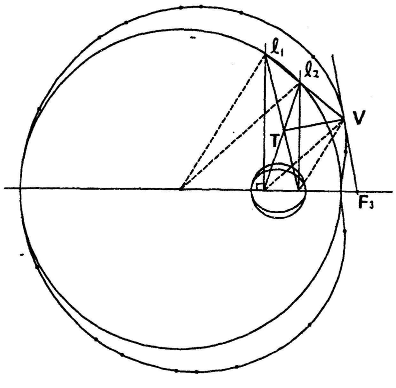

Fig. 6 Vertex of outer part

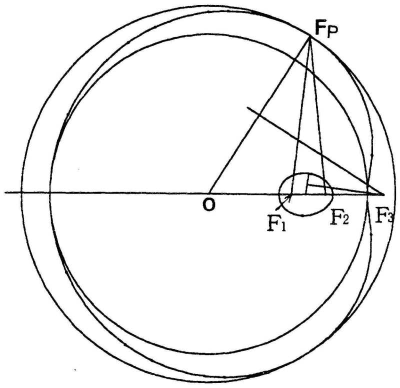

Fig. 7 Outer part between concentric circles

3.4 Outer Major Axis of Cardioid as Oval Outer Part

Cardioid has been known from old as the shape of heart, that is shown in (LOCKWOOD, 1964). It is expressed with the following equation:

$$r = ao(1-\cos\theta) \qquad (7)$$

when the origin or initial line is taken as shown in Fig. 8. By the way, the outer part of a Cartesian Oval is expressed in polar coordinates as

$$r = c\{km - n^2\cos\theta + n\sqrt{n^2\cos^2\theta - 2km\cos\theta + k^2 + m^2 - n^2}\}/(m^2-n^2) \qquad (8)$$

Here $ao = kc/(m-n)$ exists, and when we take procedure to let $n/k = e_L$ and $m/k = e_R$ approach unity, this equation agrees with Eq. (7) because of

$$r = \{1-(n/k)^2\cos\theta + (n/k)\sqrt{((n/k)\cos\theta)^2 - 2(m/k)\cos\theta + 1 + (m/k)^2 - (n/k)^2}\}(kc/(m-n))(1/(n/k+m/k))$$

Now, the length $bo = ao\sqrt{1+e_Le_R}$ of the outer major axis of the outer part in Section 3.2 becomes $ao\sqrt{2}$ when e_L and $e_R \rightarrow 1$. In short, $\sqrt{2}$ times the inscribed auxiliary circle of the outer part becomes the radius of circumscribed auxiliary circle. This means for $\theta = \pi/2$ in Eq. (7) that the longest distance from the midpoint of the symmetry axis of a Cardioid becomes $\sqrt{2}ao$ at $r = ao$ as shown in Fig. 8.

As mentioned above, this means that length of the major axis of the outer part is $\sqrt{2}$ times the length of symmetry axis at maximum in limit of $e_L = 1$ and $e_R = 1$.

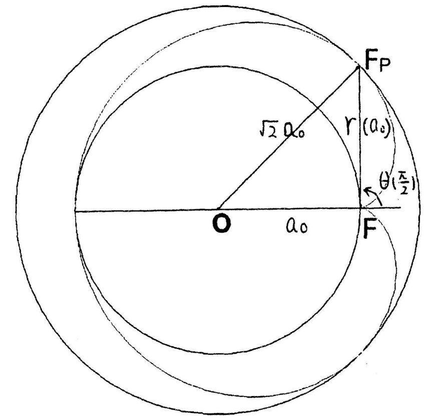

Fig. 8 Outer major axis of a Cardioid

4. AN INVARIANT AMONG SYMMETRY AND ASYMMETRY AXES OF INNER AND OUTER PART OF THE OVAL

When Cartesian Oval is expressed with Definition 1 in Section 2, inner and outer parts with the same ratio can be drawn inside and outside the circle. With Definition 2, in short, it becomes $mr_1 \pm nr_2 = kc$, resulting in the fact that it is expressed with two equations having the $\pm$ signs. These inner and outer parts have the relationship as shown in Fig. 9 in the drawing method 『To draw this oval when two circles O_{12} and O_{21} have been given as auxiliary circle』 in our paper (EBISUI, 1973). In short, when the circle O_{12} as the circumscribed circle of the inner part and the other circle O_{21} as the inscribed circle of the outer part are existent, thier two centers of similarity are F_1 and F_2. Here, the circle O_{12} is included in the circle O_{21}. Even hough Definition 3 is adapted to F_1, F_2 and the circle O_{12}, and also to F_1, F_2 and the circle O_{21}, at this time, the inner and outer parts can be obtained. Also drawn in Fig. 9 have been the inscribed circle of the inner part (auxiliary circle with inner minor axis) and the circumscribed circle of the outer part (auxiliary circle with outer major axis). From relationships between minor and major axes of the inner and between the minor axis (outer symmetry axis) and the major axis (outer major axis) of the outer part, radii of these four circles are arranged below in sequence of shorter length:

$bi = ai\sqrt{1-e_Le_R}$, ai, ao, $bo = ao\sqrt{1+e_Le_R}$

Here, from $(bi/ai)^2 = 1-e_Le_R$ and $(bo/ao)^2 = 1+e_Le_R$, the following holds true:

[Theorem] $(bi/ai)^2 + (bo/ao)^2 = 2$

$0 < bi < ai$, $ao < bo < \sqrt{2}ao$

These are equations on radii of four circles inscribing or circumscribing the Oval which is a doubly closed curve, and the equations do not contain the parameter called eccentricity, and it is a property common to whole the Cartesian Oval family to have no dependence on the eccentricity. Accordingly, this equation may be regarded as the metric invariant for the range in which the inner and outer parts of the Cartesian Oval exist.

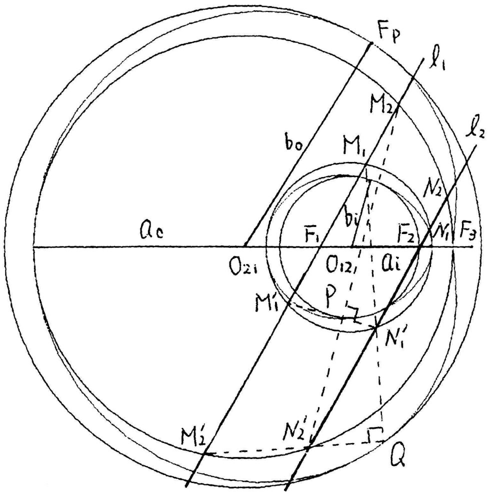

Fig. 9-1 Oval with two circles (Comprehension)

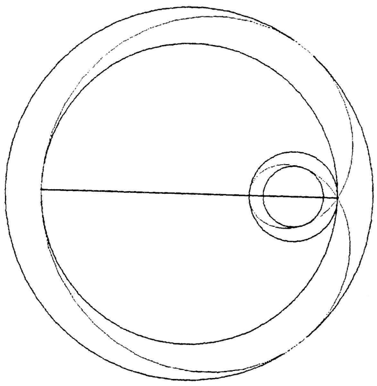

Fig. 9-2 Oval with two circles (Inscribed)

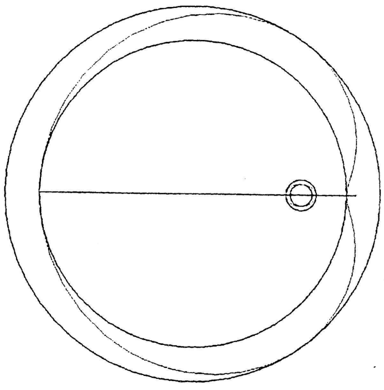

Fig. 9-3 Oval with two circles (For inner circle close to point)

5. CONCLUSION

As we have seen, the Cartesian Oval has an outer part, and its outer major axis have become known to have various properties as mentioned in Section 3. It was found in addition that the biquadratic curve called the Cartesian Oval ($mr_1 \pm nr_2 = kc$ is converted to

$$[m^2(x^2+y^2)+n^2\{(x-c)^2+y^2\}-k^2c^2]^2$$
$$=4m^2n^2\{(x-c)n^2+y^2\}(x^2+y^2)$$

in xy coordinates) has inner and outer parts, and that the existing range of the curve had properties metrically common to classical geometry on the axial length as

$$\left(\frac{\textit{Inner-minor-axis}}{\textit{Inner-symmetry-axis}}\right)^2 + \left(\frac{\textit{Outer-major-axis}}{\textit{Outer-symmetry-axis}}\right)^2 = 2$$

with no regard to values of parameters k, m and n ($k>m>n>0$) for arbitrary constants.
In addtion, Cardioid is a limiting curve of Cartesian Oval, we can say.

At last, we hope those research of Oval make clear what the oval is and what the higher order curve is. And moreover, we hope the application of oval in physics will be found.

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