AN EXTENSION TO FOURTH ORDER SURFACES BY THE OVAL WITH 3 INVERSION POINTS

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ABSTRACT

Today, according to the advancement of functional graphic software programs, beautiful graphics including multidimensional polyhedrons with mathematical structures have emerged among CGs. The fractal is one type of them, and some beautiful patterns are being expressed on such displays as television and books. This paper reports geometric definitions, expressions and figures for drawing one type of them, highlighting that this type of surface-graphic has geometric structure (inversion and its center). There are still many problems to be solved, including how to obtain its nature, that is, how its cross-sectional composition has spatial relation and differential geometric features. For example, the oval surface, one of graphics to be reported here, may have some relation to biogeometry and also be related with space structure because it is a curved surface derived from the ellipse-generalized oval. And so, the essence and appilication of real-fourth-order surface may be clear and that these study may be utilized for the development or study of technological software programs such as CG in oder to create better civilization and culture in our space.

Key words: convexity surface, pseudotorus, self-crossing surface, double closed surface, center of inversion.

1. INTRODUCTION

In these days, many advanced technological software programs have been developed, allowing it relatively easy to draw curved surfaces within three-dimensional space from parametric expressions. Although curved surfaces can be easily expressed using parameter equations, very few of them indicate the relation between their expressions and surface positions. This paper reports a derived closed-surface to be called the "segment convexity surface" which can be defined from a plane figure as is the case with a previously-reported ovaloid.

These curved surfaces include the double closed surface whose inner part is ovaloid, the fourth order surface with the same shape of torus and the self-crossing surface. They are those surfaces which can be defined from double circles with one contained in the other and their positions can be figured out as properties unattainable only with parametric expressions.

2. SEGMENT CONVEXITY SURFACE

If two points P, Q exist on a radius vector circulating on a plane with the origin F, when a circle with the mid-point of segment PQ as center and the length of PQ as diameter is vertically erected on the plane to move the radius vector, the surface of produced solid model becomes a curved one. As shown in fig. 1,fig.4, the curved surface as generated in this way shall be referred to as "segment convexity surface". Apparently, a curved surface obtained from a segment with the constant length fixed on a radius vector is a torus, and if the mid-point of PQ is F and the length of PQ is constant, a spherical surface with PQ as diameter is produced.

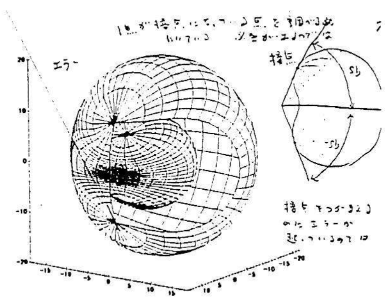

Fig. 1 Segment convexity surface

3. RELATION BETWEEN A RADIUS VECTER AND A SEGMENT FOR DRAWING AN OVAL (plane figure)

The oval as an extended type of ellipse can be defined as a curved line with a constant ratio of the distance from one point to that from a fixed circle (Ebisui, 1994).

Here, the figure to define the ratio of the oval line's inner and outer branches specified by two fixed circles (Ebisui, 1996a) shall be considered. From this figure, the spacial surface can be defined with the movement of segment on the radius vector.

3.1 Definition of oval line (construction method)

[Construction method]
The oval line as an extended type of ellipse can be uniquely defined by specifying two circles with one contained in the other.

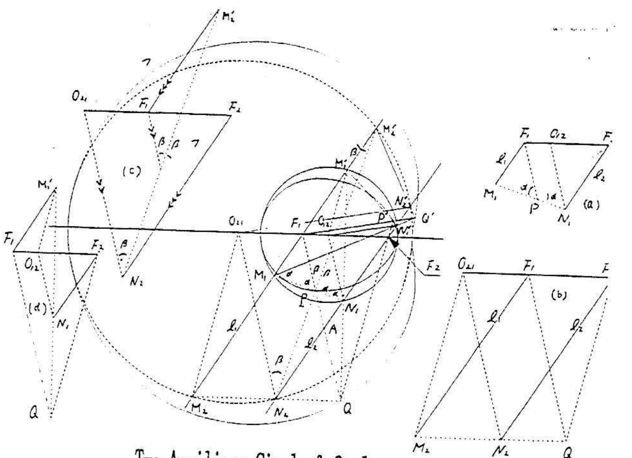

Fig.2 Two auxiliary Circles & the Oval

In fig. 2, two circles shall be specified with points O_{12} and O_{21} as centers. Centers of similitude for these two circles (see figure 2) shall be referred to as F_1, F_2. Two parallel lines passing F_1 and F_2 shall be called l_1 and l_2. Nodes of l_1 and l_2 with circle O_{12} shall be called M_1, M_1'; N_1, N_1', respectively and nodes of l_1 and l_2 with circle O_{21} shall be referred to as M_2, M_2'; N_2, N_2', respectively. The node of two lines passing F_2, F_1 as well as being parallel to two segments $O_{12}M_1$, $O_{12}N_1$, radii of circle O_{12}, respectively shall be called P as shown in fig.(a) extracted from a part of fig. 2. Likewise, Q of (b) in fig. 2 shall be obtained. At this time, when l_1, l_2 rotate around rotation centers F_1, F_2, respectively, inner and outer parts of oval line are drawn by P, Q.

3.2 Properties I of auxiliary lines for construction

Prop.(1) From Pappus' theorems(Palej,1997), M_1, P, N_1 are collinear and M_2, N_2, Q are also collinear.

Prop.(2) Lines M_1N_1 and $M_2'N_2$ are orthogonalized on point P. It is because $(\alpha+\beta)=90°$ can be obtained from $2(\alpha+\beta)=180°$.

Lines M_2N_2 and $M_1'N_1$ are also orthogonalized on point Q.

Prop.(3) Three points F_1, P, Q are collinear.
[Proof]
Since F_2 is the center of similitude, $O_{12}N_1//O_{21}N_2$ is valid. From the condition, $O_{12}N_1//F_1P$ can be obtained. From the condition, $O_{12}N_1//F_1P$, $O_{21}N_2//F_1Q$ is acquired. Therefore, $F_1P//F_1Q$ is obtained.

Likewise, three points F_1, P', Q' are collinear.

3.3 Proof that an oval line is drawn by P, Q.

In fig. 2,If the node of F_1P and l_2 is referred to as A, F_1A is constant because $F_1A//O_{12}N_1$ is valid and the radius of circle O_{12} is constant. That is, when l_2 rotates around F_2, a fixed circle (F_1 as center and F_1A as radius) is drawn by A. Here, $F_2N_1:N_1A=F_2O_{12}:O_{12}F_1=m:n$ shall be valid. In Δ F_2PA, because a line PN_1 is the bisector of $\angle$ F_2PA, $PF_2:PA=F_2N_1:N_1A=F_2O_{12}:O_{12}F_1$ can be obtained.

Therefore, since $PF_2:PA$ is constant, P is a point with a constant ratio of the distance from a fixed circle (F_1;F_1A) to that from a point F_2. At this time, since $J=F_1A=F_1P+PA=F_1P+(n/m)F_2P$, $r1+n/mr2=J$ (J=constant) is valid in bipolar (F_1, F_2)coordinates(r1,r2).

Therefore, $mr1+nr2=mJ$ can be obtained.

This expression demonstrates that P exists on Descartes' oval line.

Likewise, for Q, $mr1-nr2=mJ$ can be obtained.

3.4 Properties II of auxiliary lines for construction

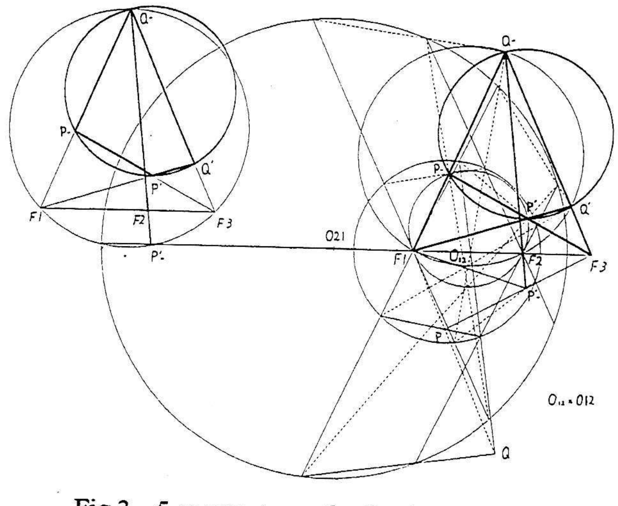

Fig.3 5 segments on the Oval

Fig. 3 shows a graphic produced by symmetrically moving a part of fig. 2 with a line F1F2 as axis of symmetry. That is, lines F1PQ and F1P-Q- are symmetric with a line F1F2.

Prop.(1) Four points F1, P-, P', F2 exist on the same circumference.
[Proof]
Since P- and P in fig. 2 and 3 are symmetric, $\angle$ F1PF2= $\angle$ M1O12N1= $\angle$ M1'O12N1'= $\angle$ F1P'F2 can be obtained, allowing $\angle$ F1P-F2= $\angle$ F1P'F2 to be true. Therefore, from the theorem of angle of circumference, the above condition is valid.

Prop.(2) Due to the above-mentioned property, two visual angles between two points F1, F2 from P-, P' are equal. Therefore, from the note (Ebisui, 1996b), a line P-P' passes the third focal point F3 in fig. 3.

Prop.(3) Four points F1, Q-, Q', F2 exist on the same circumference.
[Proof]
Since ∠ F1Q-F2=∠ F1QF2=∠ M2O21N2=∠ M2'O21N2'=∠ F1Q'F2, the above condition is valid.

Prop.(4) As the case in (2), a line Q-Q' passes the third focal point F3.

Prop.(5) Three points Q, F2, P' are collinear.
[Proof]
In fig. 2, since F1 is the center of similitude, O21M2//O12M1' is valid. From the condition, O21M2//F2Q, O12M1'//F2P' can be acquired. Then, F2Q//F2P' is obtained.

Therefore, three points Q, F2, P' are collinear.

Prop.(6) Quadrangle Q-P-P'Q' exists on the same circumference.
[Proof]
Due to the property as shown in (5) and the symmetric relation of Q-, Q with F1F2, the following equation is valid:

∠ Q-F2F1=∠ P'F2F3 ··· i)
Because of the Prop.(1) ∠ F1P-P'=∠ P'F2F3 ··· ii)
Due to the Prop.(3) ∠ Q-Q'F1=∠ Q-F2F1 ··· iii)

From conditions i), ii) and iii), ∠ F1P-P'=∠ Q-Q'F1 is valid. Therefore, a quadrangle Q-P-P'Q' exists on the same circumference.

Prop.(7) Quadrangle P'F2F3Q' exists on the same circumference.
[Proof]
From The Prop.(1), (3), (6) and Clifford's theorem(Iwata,1987), the above Prop. is valid.

4. SEGMENT CONVEXITY SURFACE DERIVED FROM THE COMPOSITION OF OVAL EXTENDED ELLIPSE

The segment convexity surface shall be considered from the graphic with P-, Q-, P', Q', F1, F2, F3 being collinear as shown in fig. 3 of section 3. So, we will obtain the equations in following subsections from fig.4

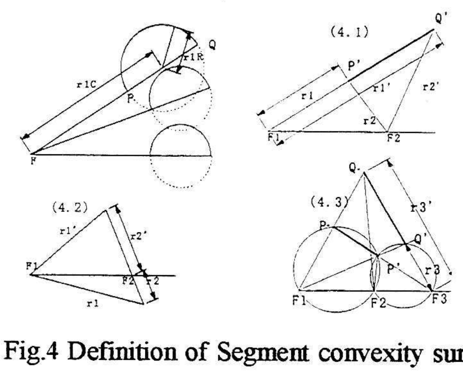

Fig.4 Definition of Segment convexity surface

4.1 Curved surface derived from a segment P'Q' on the radius vector passing the first focal point F1

With the length and position of segment P'Q' on the radius vector F1P'Q', points P', Q' exist on the oval line whose coordinates are defined as shown in fig. 4. The curved surface produced by the segment P'Q' (or P-Q-) on the radius vector passing the first focal point F1 is as follows:

i) $mr1 \pm nr2 = kc$ $(k>m>n>0)$ Definition of bipolar coordinate (c is the distance between bipole.)
ii) $r2^2=c^2+r1^2-2cr1\cdot\cos\theta$ Theorem of cosines
The segment (2·r1R) generated by radius vectors r1, r1' satisfying i) and ii) and its mid-point can be obtained as follows:
From i), $n^2 r2^2=(kc - mr1)^2$ is valid. Inserting $r2^2$ of ii) into this equation to sort out a term r1, the following equation can be obtained:
iii) $(m^2-n^2)r1^2-2(km-n^2\cos\theta)c\cdot r1+(k^2-n^2)c^2=0$
From two solutions r1, r1' of iii),and from the relation of solution and coefficient, the following expressions (product and sum) can be acquired:

$$r1\cdot r1'=(k^2-n^2)c^2/(m^2-n^2)$$
$$r1C=(r1+r1')/2=(km-n^2\cos\theta)c/(m^2-n^2)$$
$$r1R=\mathrm{abs}(r1-r1')/2=\mathrm{sqrt}((r1C)^2-r1\cdot r1')$$
$$=\sqrt{\frac{(k\,m-n^2\cos(s))^2\,c^2}{(m^2-n^2)^2}-\frac{(k^2-n^2)\,c^2}{m^2-n^2}}$$

It is used s insteed of θ in the above and following expressions.
Therefore, since the segment convexity surface in
cludes points on the circle with r1C as the distance between origin and the center of the circle and r1R as radius, it can be expressed as follows:
x:=r1C*cos(s)-r1R*cos(t)*cos(s):
y:=r1R*sin(t):
z:=r1C*sin(s)-r1R*cos(t)*sin(s):
Here,Origin names F1.
This means that x, y, z are parametric expressions of t and s if the above-indicated equations are used with r1C and r1R.

Graphical indication of (x, y, z) with Maple V is shown in fig. 5. It is a pseudo-torus.

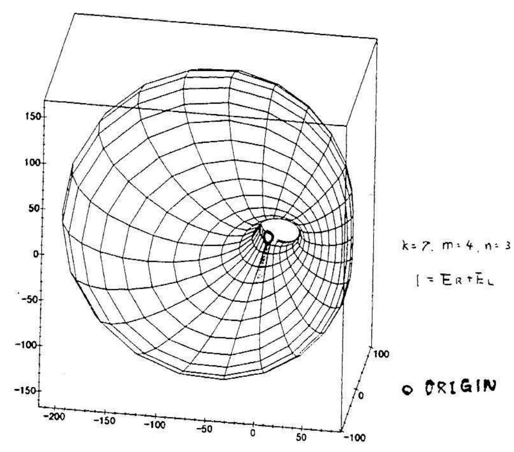

Fig.5 Pseudo-torus with a Inverse Point

5. PROPERTIES OF CONVEXITY SURFACE

5.1 Four order surface

In section 4, the following-shaped parametric expressions were generally discussed:

$x = C\cos s - R(\cos t)(\cos s)$
$y = R\sin t$
$z = C\sin s - R(\cos t)(\sin s)$
$R^2 = C^2 - K$

where, C is the first order expression of (cos s) and K is a constant.

From the above equations,useing these relations ($\cos s = x/(\sqrt{(x^2+z^2)})$: $\tan s = z/x$) and erasing s and t results in the following expressions:

$$(R\cos t\cos s)^2 + (R\cos t\sin s)^2 + (R\sin t)^2$$
$$= (C\cos s - x)^2 + (C\sin s - z)^2 + y^2$$
$$R^2 = C^2 - 2C(x\cos s + z\sin s) + x^2 + y^2 + z^2$$
$$2C(\sqrt{(x^2+z^2)}) = K + x^2 + y^2 + z^2$$

Here, uppercase C and K shall be inserted for the case of section 4.2:

$$2(A(\cos s) + B)(\sqrt{(x^2+z^2)}) = K + x^2 + y^2 + z^2$$
$$A = -m\text{^}2 * c/(m\text{^}2 - n\text{^}2) : B = k * n * c/(m\text{^}2 - n\text{^}2)$$

Both members shall be squared to remove $\sqrt{\ }$:

$$e: 4\frac{k^2 n^2 (x^2+z^2) c^2}{(m^2-n^2)^2} = \left(x^2+y^2+z^2+2\frac{m^2 c x}{m^2-n^2}+\frac{(m^2-k^2)c^2}{m^2-n^2}\right)^2$$

Here is the fourth order algeblic expression by $(x^2)^2 = x^4, y^4, z^4$ of self-cross surface. For other fourth order expressions in section 4.1 and 4.3, the same computation shall be performed.

5.2 The convexity surface has a center of inverse.

Generally, in plane figures, if $r1r1' = r2r2' = d^2$ is valid, where d is a constant in Theorem of power as shown in fig.8, r and r' have inverse relation to each other.

The convexity surface is the curved surface whose symmetric section includes the composition of inner and outer branches of oval line as discussed in section 3 and which is derived from this composition.

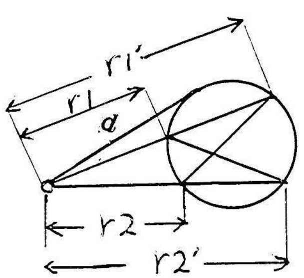

Fig. 8 Inversion

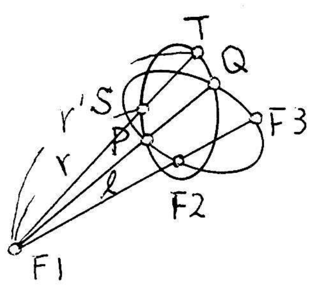

Fig. 9 Space Inversion

In this composition, two nodes of inner and outer branches of oval line with a radius vector passing its focal point F1 (similar for F2 and F3) have inverse relation to each other with the focal point as center of inversion (Lockwood, 1964).

In fig. 9, F1P・F1Q=F1F2・F1F3=constant is true. For points S, T as nodes of a line l passing a focal point F1 with convexity circle, F1S・F1T=F1P・F1Q=F1F2・F1F3=constant is valid and S, T are inverse with F1. Therefore, in the pseudotorus as discussed in section 4, nodes of any line passing F1 with this curved surface have a center of inversion F1 so that they have inverse relation to each other. For the self-crossing and double closed surfaces, F2 and F3 are centers of inversion, respectively.

6. CONCLUSION

Conversely following the above-mentioned deduction, a circle is made using a segment of a radius vector on a plane as diameter so that it has a center of inversion, to make a convexity surface defined as its locus. As the result, fourth order torus may be extended and double closed surface and self-cross surface may make the structure of fourth order convexity curved surface more clear , and these study may be utilized for the development or study of technological software programs such as CG in oder to create better culture and civilization.Thanks a lot!

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ABOUT AUTHER

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$-kr_2+mr_3=\pm n\left(\frac{k^2-m^2}{m^2-n^2}\right)c$

4.2 Curved surface derived from a segment Q-P'- on the radius vector passing the second focal point F2

Points Q-, P'- exist on the oval line whose coordinates are defined as shown in fig. 4. Let origin be F2. In the same way as section 4.1, the following equations are obtained:

mr1±nr2=kc

$r1^2=c^2+r2^2-2cr2\cdot\cos(s)$

From the above expressions, r1 is erased:

r2r2' (=K) = (m^2-k^2) * (c^2) / (m^2-n^2)

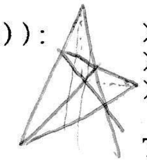
r2C:= (k*n-m^2*cos (s)) *c/ (m^2-n^2) :

r2R:=sqrt (r2C^2- (m^2-k^2) * (c^2) / (m^2-n^2)) :

x:=r2C*cos (s) -r2R*cos (t) *cos (s) :

y:=r2R*sin (t) :

z:=r2C*sin (s) -r2R*cos (t) *sin (s) :



The result of drawing this with Maple V is shown in fig. 6. It is a self-crossing surface, whose part is opened as a window.

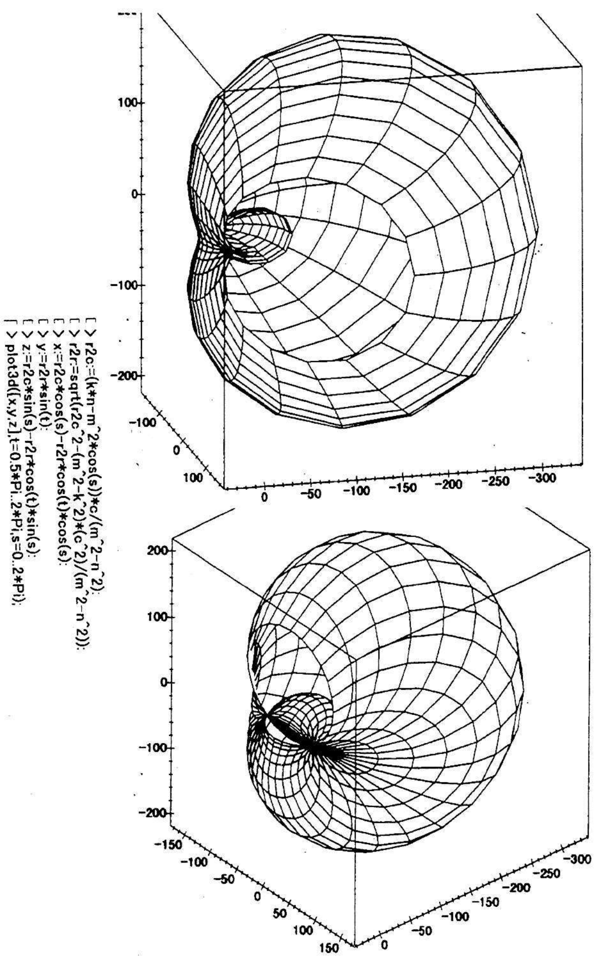


Fig. 6 Self-crossing surface

4.3 Curved surface derived from segments P-P'- and Q-Q' on the radius vector passing the third focal point F3

With P-, P', in the same way as section 4.1, the following equations are obtained from fig. 4:

i) kr1+nr3=mc'

$r1^2=c'^2+r3^2-2c'r3\cdot\cos(s)$

c'= (k^2-n^2) *c/ (m^2-n^2)

∵ c' is the distance between F1 and F3

For i) expression, please refer to the reference (Ebisui, 1996b). r1 shall be erased:

here c' is notated as c3

c3:= (k^2-n^2) *c/ (m^2-n^2) :

r3C:= (k^2*cos (s) -m*n) *c3/ (k^2-n^2) :

r3R:=sqrt (r3C^2- (k^2-m^2) * (c3^2) / (k^2-n^2)) :

> x:=r3C*cos (s) -r3R*cos (t) *cos (s) :

> y:=r3R*sin (t) :

> z:=r3C*sin (s) -r3R*cos (t) *sin (s) :

The above expression indicated the inner branch of P-, P'. As for the outer branch of Q-, Q', n can be replaced with -n in this formula.

The range of θ : (s) for (x, y, z) in the above expression shall be obtained as follows:

mr1±nr2=kc - - - - -> kr1-nr3=mc'

$r1^2=c'^2+r3^2-2c'r3\cdot\cos(s)$ c' is same above.

$r3^2=c'^2+r1^2-2c'r1\cdot\cos(s')$

cos (s') =c/ (kc/m) =m/k

From the above equations, r1, r3, s' shall be erased:

cos (s) = (sqrt ((k^2-m^2) * (k^2-n^2)) +m*n) / (k^2)

ss:=arccos ((sqrt ((k^2-m^2) * (k^2-n^2)) +m*n) / (k^2))

-ss≦ s ≦ss

Here, the expression of inner branch is m*n, that of outer branch is -m*n.

θ ' is the angle made by the radius vector of vertex in "Theorem that the tangent to vertex of oval line passes the third focal point (Ebisui, 1976)" and its initial line.

Fig. 7 indicates a double closed surface, whose part is opened as a window.

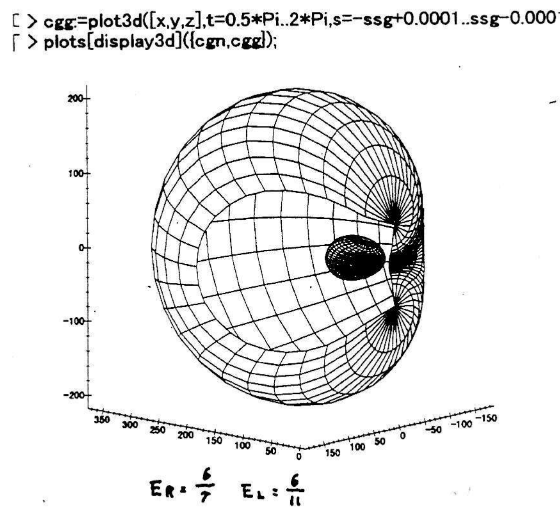


Fig. 7 Double closed surface

AN EXTENSION TO FOURTH ORDER SURFACES BY THE OVAL WITH 3 INVERSION POINTS*

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APPENDIX

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> FX:=c/(m^2-n^2)*(k^2-n^2-(k^2*cos(s)-m*n)*cos(s)+sqrt((k^2*cos(s)-m*n)^2-(k^2-m^2)*(k^2-n^2))*cos(t)*cos(s));
```

$$FX := c\left(k^2 - n^2 - (k^2\cos(s) - m\,n)\cos(s) + \sqrt{(k^2\cos(s) - m\,n)^2 - (k^2 - m^2)(k^2 - n^2)}\,\cos(t)\cos(s)\right)/(m^2 - n^2)$$

```
> FX :=
c*(k^2-n^2-(k^2*cos(s)-m*n)*cos(s)+sqrt((k^2*cos(s)-m*n)^2-(k^2-m^2)*(k^2-n^2))*cos(t)*cos(s))/(m^2-n^2);
```

$$FX := c\left(k^2 - n^2 - (k^2\cos(s) - m\,n)\cos(s) + \sqrt{(k^2\cos(s) - m\,n)^2 - (k^2 - m^2)(k^2 - n^2)}\,\cos(t)\cos(s)\right)/(m^2 - n^2)$$

```
> FY:=c*sqrt((k^2*cos(s)-m*n)^2-(k^2-m^2)*(k^2-n^2))*sin(t)/(m^2-n^2);
```

$$FY := \frac{c\sqrt{(k^2\cos(s) - m\,n)^2 - (k^2 - m^2)(k^2 - n^2)}\,\sin(t)}{m^2 - n^2}$$

```
> FZ:=c*(k^2*cos(s)-m*n-sqrt((k^2*cos(s)-m*n)^2-(k^2-m^2)*(k^2-n^2))*cos(t))*sin(s)/(m^2-n^2);
```

$$FZ := c\left(k^2\cos(s) - m\,n - \sqrt{(k^2\cos(s) - m\,n)^2 - (k^2 - m^2)(k^2 - n^2)}\,\cos(t)\right)\sin(s)/(m^2 - n^2)$$

```
> ax:=subs(c=10,k=9,m=7,n=3,FX);
```

$$ax := 18 - \frac{1}{4}(81\cos(s) - 21)\cos(s) + \frac{1}{4}\sqrt{(81\cos(s) - 21)^2 - 2304}\,\cos(t)\cos(s)$$

```
> ay:=subs(c=10,k=9,m=7,n=3,FY);
```

$$ay := \frac{1}{4}\sqrt{(81\cos(s) - 21)^2 - 2304}\,\sin(t)$$

```
> az:=subs(c=10,k=9,m=7,n=3,FZ);
```

$$az := \frac{1}{4}\left(81\cos(s) - 21 - \sqrt{(81\cos(s) - 21)^2 - 2304}\,\cos(t)\right)\sin(s)$$

```
> ss:=((m*n+sqrt((k^2-m^2)*(k^2-n^2)))/k^2);
```

$$ss := \frac{m\,n + \sqrt{(k^2 - m^2)(k^2 - n^2)}}{k^2}$$

* The main body of this paper can be found on pp. 428-432 of these Proceedings (Vol. 2).

```
> ssa:=subs(k=9,m=7,n=3,ss);
```

$$ssa := \frac{7}{27} + \frac{1}{81}\sqrt{2304}$$

```
> sa:=arccos(ssa);
```

$$sa := \arccos\left(\frac{7}{27} + \frac{1}{81}\sqrt{2304}\right)$$

```
> c1:=plot3d([ax,ay,az],s=-sa..sa,t=0..2*Pi):
>
> bx:=subs(c=10,k=9,m=7,n=-3,FX);
```

$bx :=$

$$18 - \frac{1}{4}(81\cos(s) + 21)\cos(s) + \frac{1}{4}\sqrt{(81\cos(s)+21)^2 - 2304}\cos(t)\cos(s)$$

```
> bby:=subs(c=10,k=9,m=7,n=-3,FY);
```

$$bby := \frac{1}{4}\sqrt{(81\cos(s)+21)^2 - 2304}\sin(t)$$

```
> bz:=subs(c=10,k=9,m=7,n=-3,FZ);
```

$$bz := \frac{1}{4}\left(81\cos(s) + 21 - \sqrt{(81\cos(s)+21)^2 - 2304}\cos(t)\right)\sin(s)$$

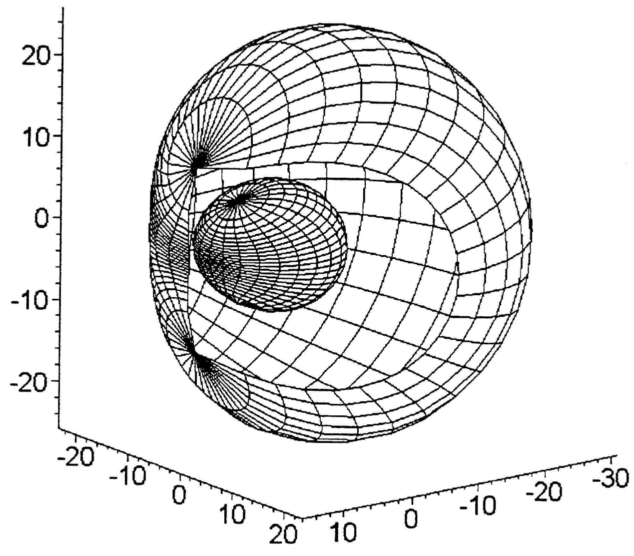
```
>
> ssb:=subs(k=9,m=7,n=-3,ss);
```

$$ssb := -\frac{7}{27} + \frac{1}{81}\sqrt{2304}$$

```
> sb:=evalf(arccos(ssb),30);
```

$$sb := 1.23095941734077468213492917825$$

```
> #c2:=plot3d([bx,bby,bz],s=-sb+0.00001..sb-0.00001,t=0.5*Pi..2*Pi):
> c2:=plot3d([bx,bby,bz],s=-sb..sb,t=0.5*Pi..2*Pi):
> plots[display3d]({c1,c2});
```



$$> \frac{154}{9} - \frac{2}{9}(81\cos(s)+14)\cos(s) + \frac{2}{9}\sqrt{(81\cos(s)+14)^2-2464}\,\cos(t)\cos(s)$$

```
> bby:=subs(c=10,k=9,m=7,n=-2,FY);
```

$$bby := \frac{2}{9}\sqrt{(81\cos(s)+14)^2-2464}\,\sin(t)$$

```
> bz:=subs(c=10,k=9,m=7,n=-2,FZ);
```

$$bz := \frac{2}{9}\left(81\cos(s)+14-\sqrt{(81\cos(s)+14)^2-2464}\,\cos(t)\right)\sin(s)$$

```
>
> ssb:=subs(k=9,m=7,n=-2,ss);
```

$$ssb := -\frac{14}{81}+\frac{1}{81}\sqrt{2464}$$

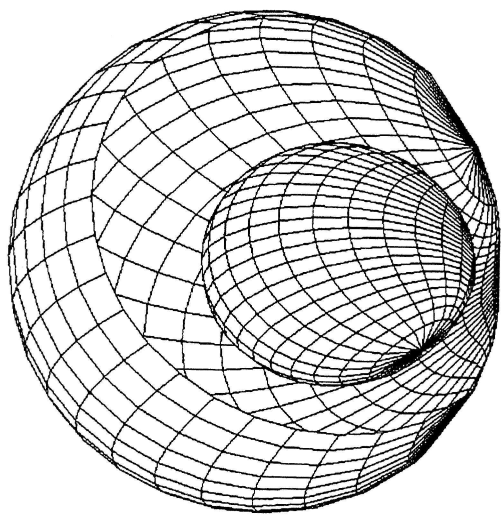
```
> sb:=evalf(arccos(ssb),30);
```

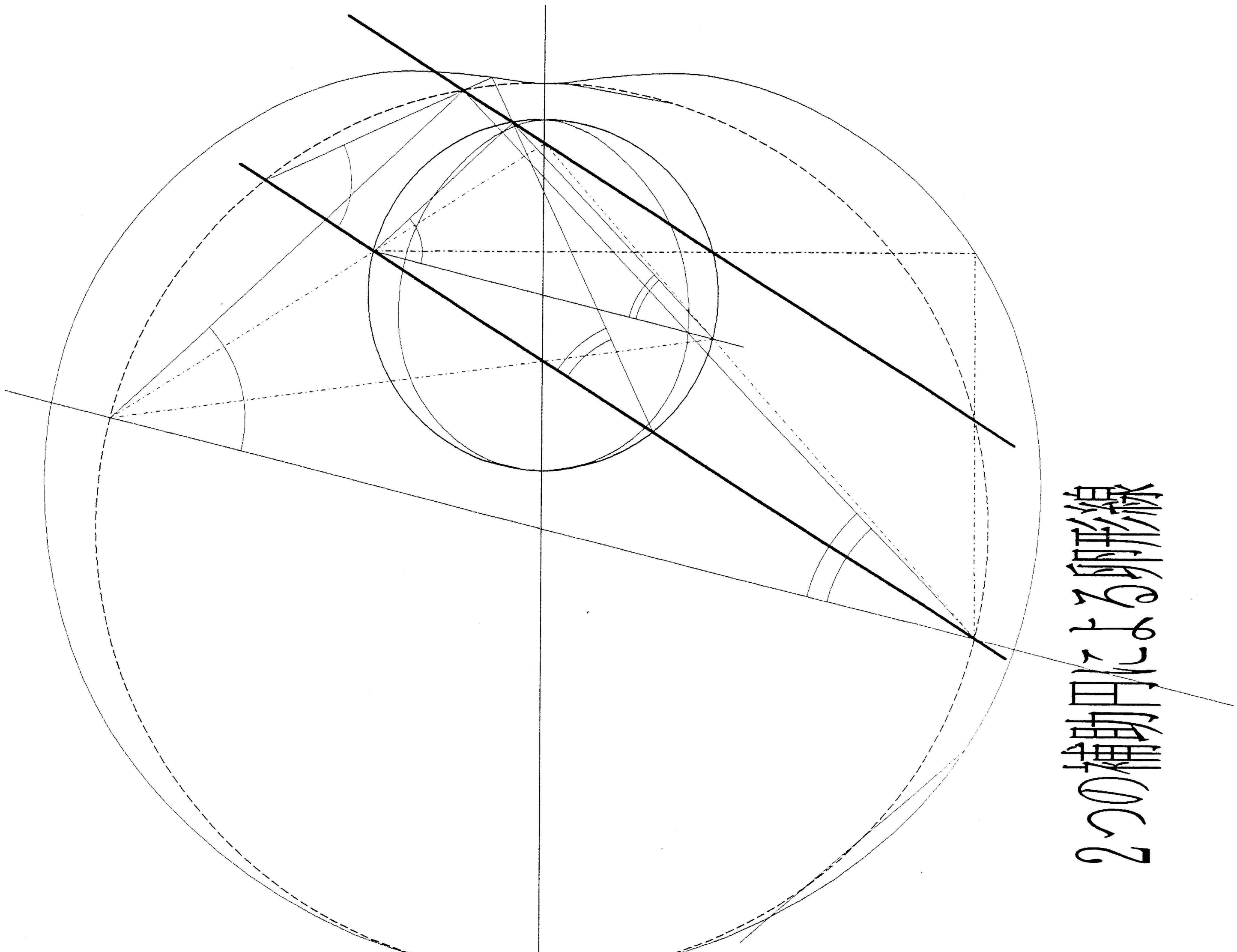
$$sb := 1.11521560018802359745160319564$$

```
> #c2:=plot3d([bx,bby,bz],s=-sb+0.00001..sb-0.00001,t=0.5*Pi..2*Pi):
> c2:=plot3d([bx,bby,bz],s=-sb..sb,t=0.75*Pi..2*Pi):
> plots[display3d]({c1,c2});
```

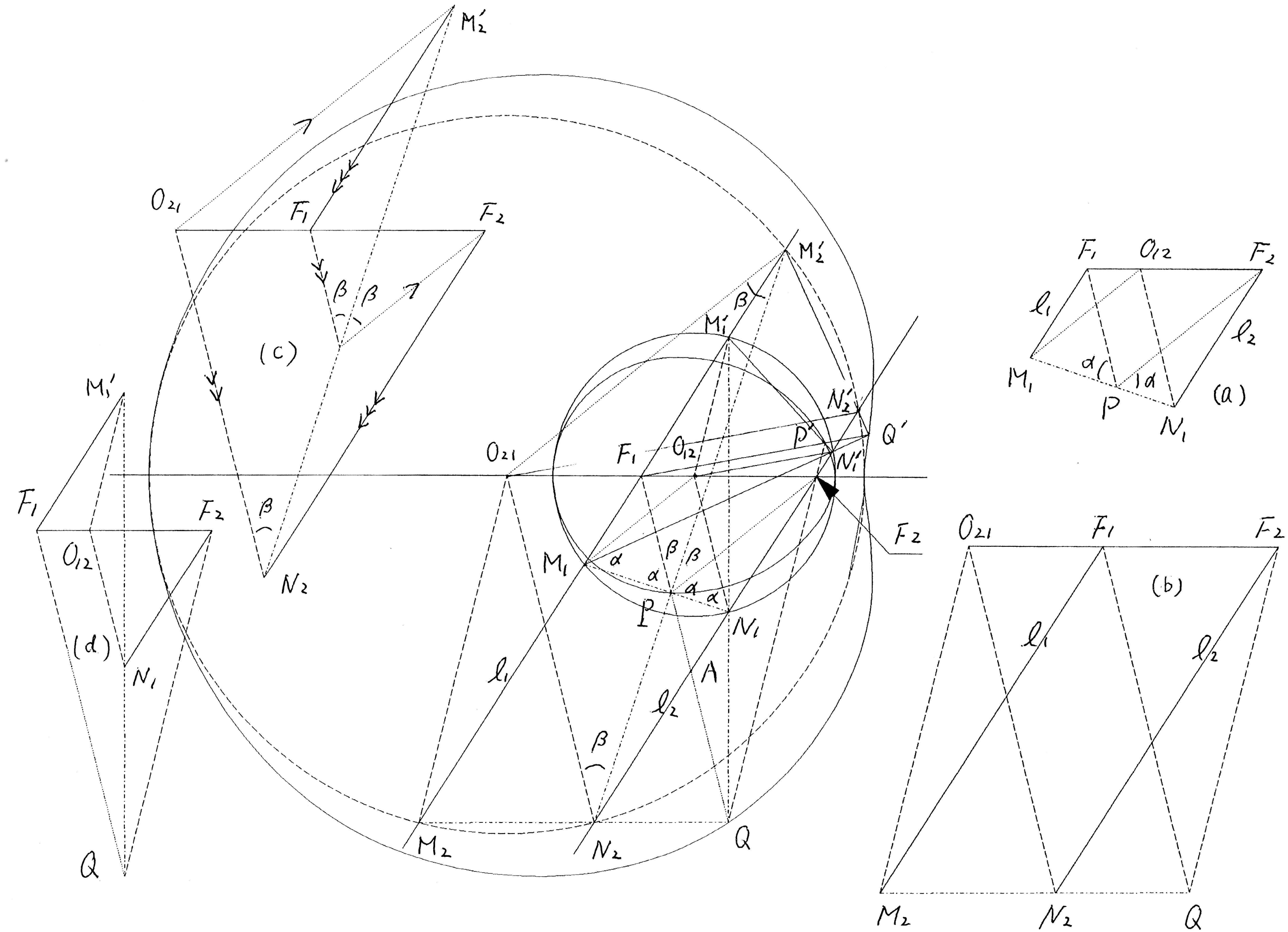
$\theta = -87$

$\varphi = -73$





2つの補助円による卵形線



Two Auxiliary Circle & Oval