デカルトの卵形線の周長

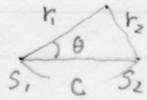
卵形線の双極座標による定義式は

$$m r_1 \pm n r_2 = kc \quad \cdots ①$$

ただし $k > m > n > 0$　c：極間キョリ
である任意定数

閉曲線の周長の極座標による公式

$$l = 2\int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \quad \cdots ②$$



①において 第1極を極とする極座標表示では

$$r_2^2 = r_1^2 + c^2 - 2r_1 c\cos\theta.$$

$$\therefore r_2 = \pm\sqrt{r_1^2 + c^2 - 2r_1 c\cos\theta} \quad \cdots ③$$

①に③を代入すれば 卵形線の極座標表示.

$$m r_1 \pm n\sqrt{r_1^2 + c^2 - 2r_1 c\cos\theta} = kc$$

ここで $r_1 = r$ とおき rについて解くと.

$$(mr - kc)^2 = n^2(r^2 + c^2 - 2rc\cos\theta)$$

$$(m^2 - n^2)r^2 - 2(km - n^2\cos\theta)cr + (k^2 - n^2)c^2 = 0 \quad - ④$$

$$\therefore r = \frac{km - n^2\cos\theta \pm \sqrt{(km - n^2\cos\theta)^2 - (m^2 - n^2)(k^2 - n^2)}}{m^2 - n^2}\cdot c$$

$$= \frac{km - n^2\cos\theta \pm \sqrt{-2kmn^2\cos\theta + n^4\cos^2\theta + n^2(k^2 - n^2 + m^2)}}{m^2 - n^2}\cdot c$$

$$= \frac{km - n^2\cos\theta \pm n\sqrt{n^2\cos^2\theta - 2km\cos\theta + k^2 + m^2 - n^2}}{m^2 - n^2}\cdot c \quad ⑤$$

④をθについて微分すると.

$$2(m^2 - n^2)r\frac{dr}{d\theta} - 2(km - n^2\cos\theta)\frac{dr}{d\theta}c - 2n^2 cr\sin\theta = 0$$

$$\therefore \frac{dr}{d\theta} = \frac{n^2 cr\sin\theta}{(m^2 - n^2)r - (km - n^2\cos\theta)c}$$

$$= \frac{n^2 r\sin\theta}{\pm n\sqrt{n^2\cos^2\theta - 2km\cos\theta + k^2 + m^2 - n^2}}$$

$$\therefore \left(\frac{dr}{d\theta}\right)^2 = \frac{n^2 r^2\sin^2\theta}{(n^2\cos^2\theta - 2km\cos\theta + k^2 + m^2 - n^2)} \quad \cdots ⑥$$

∴

②⑥より

$$l = 2\int_0^\pi \sqrt{r^2 + \frac{n^2 r^2 \sin^2\theta}{(n^2\cos^2\theta - 2km\cos\theta + k^2+m^2-n^2)}}\, d\theta$$

$$= 2\int_0^\pi \sqrt{\frac{k^2+m^2-2km\cos\theta}{(n^2\cos^2\theta-2km\cos\theta+k^2+m^2-n^2)}}\, r\, d\theta$$

$$= 2\int_0^\pi \left\{\frac{c\sqrt{k^2+m^2-2km\cos\theta}\cdot(km-n^2\cos\theta)}{(m^2-n^2)\sqrt{n^2\cos^2\theta-2km\cos\theta+k^2+m^2-n^2}} \pm \frac{n\sqrt{k^2+m^2-2km\cos\theta}}{m^2-n^2}\cdot c\right\} d\theta$$

$$= 2\frac{c}{m^2-n^2}\int_0^\pi \left\{\frac{(km-n^2\cos\theta)\sqrt{k^2+m^2-2km\cos\theta}}{\sqrt{n^2\cos^2\theta-2km\cos\theta+k^2+m^2-n^2}} \pm n\sqrt{k^2+m^2-2km\cos\theta}\right\}d\theta$$

$\cos\theta = x$ とおく $-\sin\theta\, d\theta = dx$. $-\sqrt{1-x^2}\,d\theta = dx$

$$l = \frac{2c}{m^2-n^2}\int_{-1}^1 \left\{\frac{(km-n^2x)\sqrt{k^2+m^2-2kmx}}{\sqrt{n^2x^2-2kmx+k^2+m^2-n^2}} \pm n\sqrt{k^2+m^2-2kmx}\right\}\frac{dx}{\sqrt{1-x^2}}$$

$$l = \frac{2c}{m^2-n^2}\int_{-1}^1\left\{\frac{(km-n^2x)\sqrt{k^2+m^2-2kmx}}{\sqrt{1-x^2}\sqrt{n^2x^2-2kmx+k^2+m^2-n^2}} \pm \frac{n\sqrt{k^2+m^2-2kmx}}{\sqrt{1-x^2}}\right\}dx$$

$\frac{m}{k}=K_1$ $\frac{n}{k}=K_2$ とおく

$$l = \frac{2ck^2}{m^2-n^2}\int_{-1}^1 \frac{(K_1-K_2^2x)\sqrt{1+K_1^2-2K_1x}}{\sqrt{(1-x^2)(K_2^2x^2-2K_1x+1+K_1^2-K_2^2)}}\,dx$$

$$\pm \frac{2nc}{m^2-n^2}\int_{-1}^1\frac{\sqrt{k^2+m^2-2kmx}}{\sqrt{1-x^2}}\,dx.$$

$$= \frac{2c}{K_1^2-K_2^2}\int_{-1}^1 \frac{(K_1-K_2^2x)\sqrt{1+K_1^2-2K_1x}}{\sqrt{(1-x^2)(1+K_1^2-K_2^2-2K_1x+K_2^2x^2)}}\,dx$$ 次のページへ

$x=\sin\varphi$

$$\pm\frac{2nc}{m^2-n^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sqrt{k^2+m^2-2km\sin\varphi}}{\sqrt{1-\sin^2\varphi}}\cos\varphi\,d\varphi$$

$$\pm \frac{2nc}{m^2-n^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{k^2+m^2-2km\sin\varphi}\,d\varphi$$

$x=\sin\theta$

$$\pm\frac{2nc}{m^2-n^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{(k+m)^2-2km(1+\sin\varphi)}\,d\varphi$$

$$\pm\frac{2nc(k+m)}{m^2-n^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-\frac{2km}{(k+m)^2}(1+\sin\varphi)}\,d\varphi$$

$$\pm \frac{2nc}{m^2-n^2}\int_{-1}^{1}\frac{\sqrt{k^2+m^2-2kmx}}{\sqrt{1-x^2}}\,dx$$

$$= \pm \frac{2nc}{m^2-n^2}\int_{-1}^{1}\frac{k^2+m^2-2kmx}{\sqrt{(1-x^2)(k^2+m^2-2kmx)}}\,dx$$

$1+x=y^2$ とおく

$$= \pm \frac{2nc}{m^2-n^2}\int_{0}^{\sqrt{2}}\frac{\{k^2+m^2+2km(1-y^2)\}2y}{\sqrt{y^2(2-y^2)\{k^2+m^2+2km(1-y^2)\}}}\,dy$$

$$= \pm \frac{4nc}{m^2-n^2}\int_{0}^{\sqrt{2}}\frac{(k+m)^2-2kmy^2}{\sqrt{(2-y^2)\{(k+m)^2-2kmy^2\}}}\,dy$$

$$= \pm \frac{4nc}{m^2-n^2}\int_{0}^{\sqrt{2}}\frac{\sqrt{(k+m)^2-2kmy^2}}{\sqrt{2-y^2}}\,dy$$

$y=\sqrt{2}x$ とおく

$$= \pm \frac{4nc}{m^2-n^2}\int_{0}^{1}\frac{\sqrt{(k+m)^2-4kmx^2}}{\sqrt{1-x^2}}\,dx$$

$$= \pm \frac{4nc(k+m)}{m^2-n^2}\int_{0}^{1}\frac{\sqrt{1-\frac{4km}{(k+m)^2}x^2}}{\sqrt{1-x^2}}\,dx$$

$$= \pm \frac{4cK_2(1+K_1)}{K_1^2-K_2^2}\int_{0}^{1}\sqrt{\frac{1-\frac{4K_1}{(1+K_1)^2}x^2}{1-x^2}}\,dx$$

$$= \pm \frac{4(1+K_1)K_2c}{K_1^2-K_2^2}\,E\left(\frac{2\sqrt{K_1}}{1+K_1}\right)$$

$$\int_{0}^{1}\sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = E(k)$$

$$l = \frac{2c}{K_1^2-K_2^2}\left\{\int_{-1}^{1}\frac{(K_1-K_2^2x)\sqrt{1+K_1^2-2K_1x}}{\sqrt{(1-x^2)(1+K_1^2+K_2^2-2K_1x+K_2^2x^2)}}\,dx \pm 2(1+K_1)K_2\,E\left(\frac{2\sqrt{K_1}}{1+K_1}\right)\right\}$$

$E(x)$ は 第2種完全楕円積分

2003-7-26 付加

$$l^* = \frac{4c}{K_1^2-K_2^2}\int_{-1}^{1}\frac{(K_1-K_2^2x)\sqrt{1+K_1^2-2K_1x}}{\sqrt{(1-x^2)(1+K_1^2+K_2^2-2K_1x+K_2^2x^2)}}\,dx$$